UNIT-V

Deflection of Beams

Introduction

The deformation of a beam is usually expressed in terms of its deflection from its original unloaded position. The deflection is measured from the original neutral surface of the beam to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam.

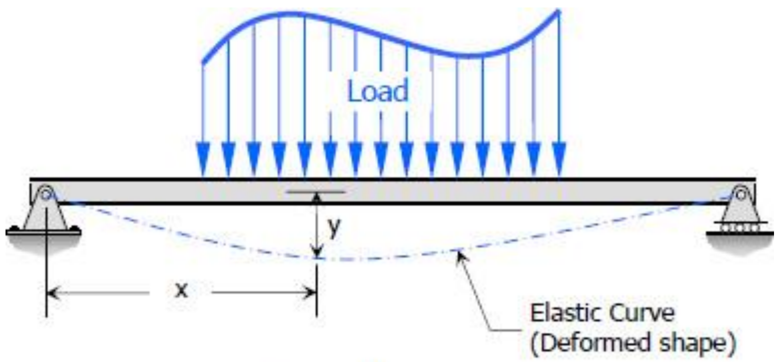


Figure: Elastic curve

A. Methods of Determining Beam Deflections

Methods for the determination of beam deflections include:

1. Double-Integration Method
2. Macaulay's Method
3. Moment-Area Method
4. Conjugate-beam Method
5. Strain - Energy method (Castigliano's Theorem)
6. Virtual work method

Of these methods, the first four shall be discussed in this course.

The stress, strain, dimension, curvature, elasticity, are all related, under certain assumption, by the theory of simple bending. This theory relates to beam flexure resulting from couples applied to the beam without consideration of the shearing forces.

B. Superposition Principle

The superposition principle is one of the most important tools for solving beam loading problems allowing simplification of very complicated design problems.

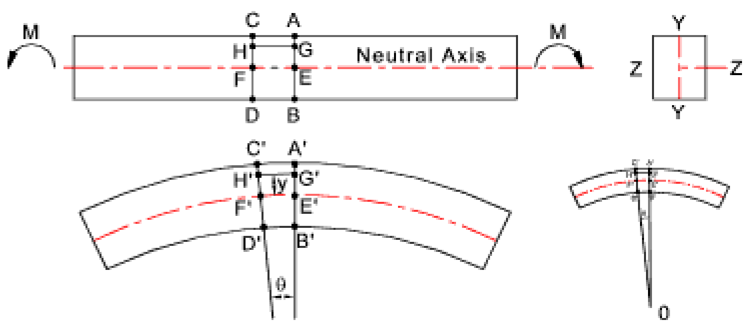
For beams subjected to several loads of different types the resulting shear force, bending moment, slope and deflection can be found at any location by summing the effects due to each load acting separately to the other loads.

C. Nomenclature

e = strain
E = Young's Modulus = σ /e (N/m^2)
y = distance of surface from neutral surface (m).
R = Radius of neutral axis (m).
I = Moment of Inertia (m^4 - more normally cm^4)
Z = section modulus = I/y_{max}(m^3 - more normally cm^3)
F = Force (N)
x = Distance along beam
δ = deflection (m)
θ = Slope (radians)
σ = stress (N/m^2)

D. Review of Simple Bending

A straight bar of homogeneous material is subject to only a moment at one end and an equal and opposite moment at the other end...



Assumptions

The beam is symmetrical about Y-Y. The traverse plane sections remain plane and normal to the longitudinal fibres after bending (Beroulli's assumption). The fixed relationship between stress and strain (Young's Modulus)for the beam material is the same for tension and compression (σ= E.e)

Consider two section very close together (AB and CD).
After bending the sections will be at A'B' and C'D' and are no longer parallel. AC will have extended to A'C' and BD will have compressed to B'D'
The line EF will be located such that it will not change in length. This surface is called neutral surface and its intersection with Z_Z is called the neutral axis
The development lines of A'B' and C'D' intersect at a point 0 at an angle of θ radians and the radius of E'F' = R
Let y be the distance(E'G') of any layer H'G' originally parallel to EF..Then

$$H'G'/E'F' =(R+y)\theta /R \theta = (R+y)/R$$

And the strain e at layer H'G' =

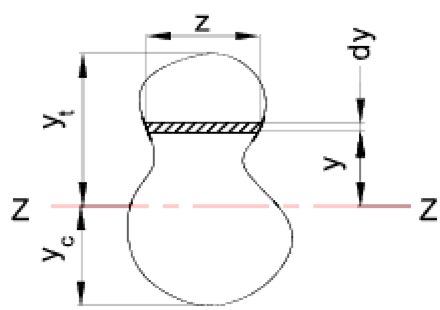
$$e = (H'G' - HG) / HG = (H'G' - HG) / EF = [(R+y)\theta - R\theta] / R\theta = y / R$$

The accepted relationship between stress and strain is $\sigma = E.e$ Therefore

$$\sigma = E.e = E.\ y / R$$
$$\sigma / E = y / R$$

Therefore, for the illustrated example, the tensile stress is directly related to the distance above the neutral axis. The compressive stress is also directly related to the distance below the neutral axis. Assuming E is the same for compression and tension the relationship is the same.

As the beam is in static equilibrium and is only subject to moments (no vertical shear forces) the forces across the section (AB) are entirely longitudinal and the total compressive forces must balance the total tensile forces. The internal couple resulting from the sum of (σ.dA .y) over the whole section must equal the externally applied moment.



$$\sum(\sigma.\delta A) = 0 \quad \text{therefore} \quad \sum(\sigma.z\ \delta y) = 0$$

$$\text{As} \quad \sigma = \frac{yE}{R} \quad \text{therefore} \quad \frac{E}{R}\sum(y.\ \delta A) = 0 \quad \text{and} \quad \frac{E}{R}\sum(y.\ z\delta y) = 0$$

This can only be correct if $\Sigma(y\delta a)$ or $\Sigma(y.z.\delta y)$ is the moment of area of the section about the neutral axis. This can only be zero if the axis passes through the centre of gravity (centroid) of the section.

The internal couple resulting from the sum of (σ.dA .y) over the whole section must equal the externally applied moment. Therefore the couple of the force resulting from the stress on each area when totalled over the whole area will equal the applied moment

The force on each area element = $\sigma.\ \delta A = \sigma.\ z.\ \delta y$

The resulting moment = $y.\sigma.\ \delta A = \sigma.\ z.\ y.\ \delta y$

The total moment $M = \sum(y.\sigma.\delta A)$ and $\sum(\sigma.z.y\ \delta y)$

Using $\frac{E}{R} y = \sigma$

$M = \frac{E}{R}\sum(y^2.\ \delta A)$ and $\frac{E}{R}\sum(z.y^2\delta y)$

$\sum(y^2.\delta A)$ is the Moment of Inertia of the section(I)

From the above the following important simple beam bending relationship results

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

It is clear from above that a simple beam subject to bending generates a maximum stress at the surface furthest away from the neutral axis. For sections symmetrical about Z-Z the maximum compressive and tensile stress is equal.

$$\sigma_{max} = y_{max} . M / I$$

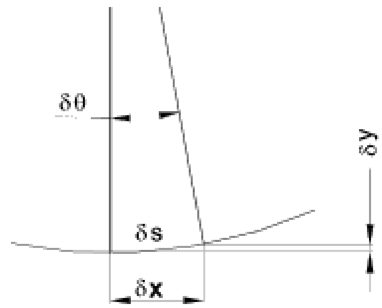
The factor I / y_{max} is given the name section Modulus (Z) and therefore

$$\sigma_{max} = M / Z$$

Values of Z are provided in the tables showing the properties of standard steel sections.

Differential Equation for the Elastic Curve

Below is shown the arc of the neutral axis of a beam subject to bending.



For small angle dy/dx = tan θ = θ
The curvature of a beam is identified as dθ /ds = 1/R
In the figure δθ is small and δx; is practically = δs; i.e ds /dx =1

$$\frac{1}{R} = \frac{d\theta}{ds} = \frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

From this simple approximation the following relationships are derived.

$$\frac{1}{R} = \frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\text{Slope} = \theta = \frac{dy}{dx} = \int\left(\frac{d^2y}{dx^2}\right)dx = \int\frac{M}{EI}\,dx$$

The deflection between limits is obtained by further integration.

$$\text{Deflection} = x = \int \theta\,dx = \int\left(\frac{dy}{dx}\right)dx = \int\int\frac{M}{EI}\,dx$$

It has been proved earlier that dM/dx = -S and dS/dx = w = - d^2M /dx^2
Where S = the shear force M is the moment and w is the distributed load /unit length of beam.
Therefore

$$S = \frac{dy}{dx}\left(EI\,\frac{d^2y}{dx^2}\right) = EI\,\frac{d^3y}{dx^3} \text{ and } -w = EI\,\frac{d^4y}{dx^4}$$

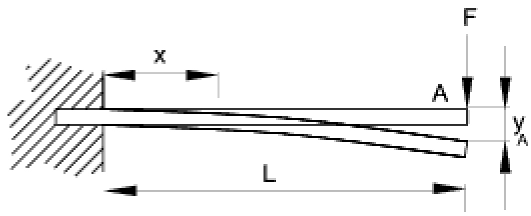
If w is constant or a integrable function of x then this relationship can be used to arrive at general expressions for S, M, dy/dx, or y by progressive integrations with a constant of integration being added at each stage. The properties of the supports or fixings may be used to determine the constants. (x= 0 - simply supported, dx/dy = 0 fixed end etc)

In a similar manner if an expression for the bending moment is known then the slope and deflection can be obtained at any point x by single and double integration of the relationship and applying suitable constants of integration of $\frac{d^2y}{dx^2} = \frac{M}{EI}$

Evaluation of deflection by double-integration method

A. Example 1- Cantilever beam

Consider a cantilever beam (uniform section) with a single concentrated load at the end. At the fixed end x = 0, dy = 0 , dy/dx = 0



From the equilibrium balance ..At the support there is a resisting moment -FL and a vertical upward force F. At any point x along the beam there is a moment F(x - L) = M_x = EI d^2y/dx^2

$$EI\frac{d^2y}{dx^2} = -F\,(L-x) \quad \text{Integrating}$$

$$EI\frac{dy}{dx} = -F\left(Lx - \frac{x^2}{2}\right) + C_1 \;.....(C_1=0 \text{ because } dy/dx = 0 \text{ at } x = 0)$$

Integrating again

$$EI\,y = -F\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2 \;.....(C=0 \text{ because } y = 0 \text{ at } x = 0)$$

$$\text{At end A } \left(\frac{dy}{dx}\right)_A = -\frac{F}{EI}\left(L^2 - \frac{L^2}{2}\right) = -\frac{FL^2}{2EI} \quad \text{and} \quad y_A = -\frac{F}{EI}\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -\frac{FL^3}{3EI}$$

Macaulay's Method / Singularity Functions

The basic equation governing the slope and deflection of beams is

$\frac{d^2y}{dx^2} = \frac{M}{EI}$, where M is a function of x. This is derived from the Euler-Bernoulli beam theory, based on the simplifying assumptions.

The method of integration of the above equation provides a convenient and effective way of determining the slope and deflection at any point of a beam, as long as the bending moment can be represented by a single analytical function M(x). However, when the loading of the beam is such that two different functions are needed to represent the bending moment over the entire length of the beam four constants of integration are required, and an equal number of equations, expressing continuity conditions at point of concentrated load, as well as boundary conditions at the supports A and B, must be used to determine these constants. If three or more functions were needed to represent the bending moment, additional constants and a corresponding number of additional equations would be required, resulting in rather lengthy computations. In this section these computations will be simplified through the use of the singularity functions. This is the Macaulay's method.

For general case of loadings, M(x), can be expressed in the form:

$$M(x) = M_1(x) + P_1\langle x - a_1\rangle + P_2\langle x - a_2\rangle + P_3\langle x - a_3\rangle + \ldots$$

where the quantity $P_i\langle x - a_i\rangle$ represents the bending moment at the section 'x' due to point load P_i located at distance a_i from the end. The quantity $\langle x - a_i\rangle$ is a Macaulay bracket defined as

$$\langle x - a_i\rangle = \begin{cases} 0 & \text{if } x < a_i \\ x - a_i & \text{if } x > a_i \end{cases}$$

Ordinarily, when integrating $P_i(x - a_i)$ we get,

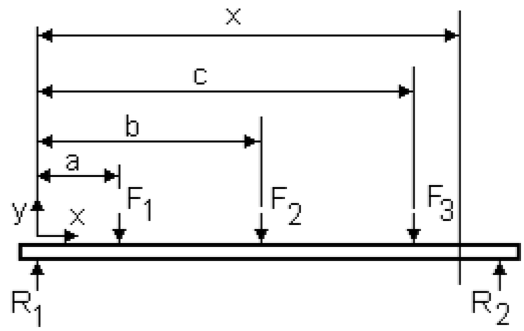
$$\int P(x-a)\ dx = P\left[\frac{x^2}{2} - ax\right] + C$$

However, when integrating expressions containing Macaulay brackets, we have to do this way:

$$\int P\langle x-a\rangle\ dx = P\frac{\langle x-a\rangle^2}{2} + C_m$$

Using these integration rules makes the calculation of the deflection of Euler-Bernoulli beams simple in situations where there are multiple point loads and point moments.

The steps for finding deflections by Macaulay's method are shown by the following example of a simply supported beam:



1. Write down the bending moment equation placing x on the extreme right hand end of the beam so that it contains all the loads. Write all terms containing x in angle brackets.

$$EI\frac{d^2y}{dx^2} = M = R_1\langle x\rangle - F_1\langle x-a\rangle - F_2\langle x-b\rangle - F_3\langle x-c\rangle$$

2. Integrate once treating the whole brackets as the variables.

$$EI\frac{dy}{dx} = R_1\frac{\langle x\rangle^2}{2} - F_1\frac{\langle x-a\rangle^2}{2} - F_2\frac{\langle x-b\rangle^2}{2} - F_3\frac{\langle x-c\rangle^2}{2} + C_1$$

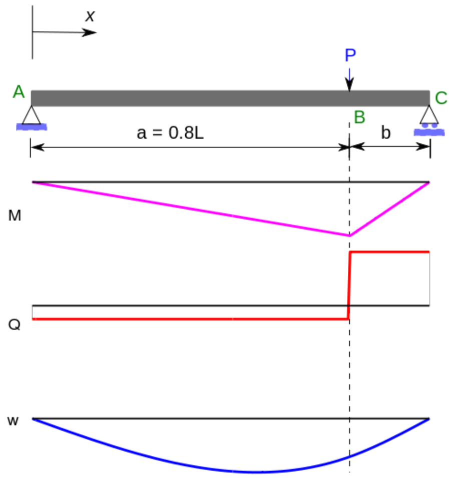
3. Integrate again using the same rules.

$$EI\ y = R_1\frac{\langle x\rangle^3}{6} - F_1\frac{\langle x-a\rangle^3}{6} - F_2\frac{\langle x-b\rangle^3}{6} - F_3\frac{\langle x-c\rangle^3}{6} + C_1x + C_2$$

4. Use boundary conditions to solve C_1 and C_2.

5. Solve slope and deflection by putting in appropriate value of x. IGNORE any brackets containing negative values.

Example 1: Simply Supported Beam with Eccentric Point Load



Consider a simply supported beam with a single eccentric concentrated load as shown in the figure.

The notations used in this example

(a) bending moment = M

(b) shear force = Q

(c) deflection = w (instead of y)

The first step is to find M. The reactions at the supports A and C are determined from the balance of forces and moments as

$$R_A + R_C = P, \quad LR_C = Pa$$

Therefore $R_A = Pb/L$ and the bending moment at a point D between A and B ($0 < x < a$) is given by

$$M = R_A x = Pbx/L$$

Using the moment-curvature relation and the Euler-Bernoulli expression for the bending moment, we have

$$EI\frac{d^2w}{dx^2} = \frac{Pbx}{L}$$

Integrating the above equation we get, for $0 < x < a$,

$$EI\frac{dw}{dx} = \frac{Pbx^2}{2L} + C_1 \qquad \text{(i)}$$

$$EIw = \frac{Pbx^3}{6L} + C_1 x + C_2 \qquad \text{(ii)}$$

At $x = a_-$

$$EI\frac{dw}{dx}(a_-) = \frac{Pba^2}{2L} + C_1 \qquad \text{(iii)}$$

$$EIw(a_-) = \frac{Pba^3}{6L} + C_1 a + C_2 \qquad \text{(iv)}$$

For a point D in the region BC ($a < x < L$), the bending moment is

$$M = R_A x - P(x-a) = Pbx/L - P(x-a)$$

In Macaulay's approach we use the Macaulay bracket form of the above expression to represent the fact that a point load has been applied at location B, i.e.,

$$M = \frac{Pbx}{L} - P\langle x-a \rangle$$

Therefore the Euler-Bernoulli beam equation for this region has the form

$$EI\frac{d^2w}{dx^2} = \frac{Pbx}{L} - P\langle x-a \rangle$$

Integrating the above equation, we get for $a < x < L$

$$EI\frac{dw}{dx} = \frac{Pbx^2}{2L} - P\frac{\langle x-a \rangle^2}{2} + D_1 \qquad \text{(v)}$$

$$EIw = \frac{Pbx^3}{6L} - P\frac{\langle x-a \rangle^3}{6} + D_1x + D_2 \qquad \text{(vi)}$$

At $x = a_+$

$$EI\frac{dw}{dx}(a_+) = \frac{Pba^2}{2L} + D_1 \qquad \text{(vii)}$$

$$EIw(a_+) = \frac{Pba^3}{6L} + D_1a + D_2 \qquad \text{(viii)}$$

Comparing equations (iii) & (vii) and (iv) & (viii) we notice that due to continuity at point B, D_1= C_1 and D_2 = C_2 . The above observation implies that for the two regions considered, though the equation for bending moment and hence for the curvature are different, the constants of integration got during successive integration of the equation for curvature for the two regions are the same.

The above argument holds true for any number/type of discontinuities in the equations for curvature, provided that in each case the equation retains the term for the subsequent region in the form $\langle x-a \rangle^n, \langle x-b \rangle^n, \langle x-c \rangle^n$ etc. It should be remembered that for any x, giving the quantities within the brackets, as in the above case, -ve should be neglected, and the calculations should be made considering only the quantities which give +ve sign for the terms within the brackets.

Reverting to the problem, we have

$$EI\frac{d^2w}{dx^2} = \frac{Pbx}{L} - P\langle x-a \rangle$$

It is obvious that the first term only is to be considered for $x < a$ and both the terms for $x > a$ and the solution is

$$EI\frac{dw}{dx} = \left[\frac{Pbx^2}{2L} + C_1\right] - \frac{P\langle x-a \rangle^2}{2}$$

$$EIw = \left[\frac{Pbx^3}{6L} + C_1x + C_2\right] - \frac{P\langle x-a \rangle^3}{6}$$

Note that the constants are placed immediately after the first term to indicate that they go with the first term when x < a and with both the terms when x > a. The Macaulay brackets help as a reminder that the quantity on the right is zero when considering points with x < a.

Boundary conditions:

As $w = 0$ at $x = 0$, $C2 = 0$. Also, as $w = 0$ at $x = L$,

$$\left[\frac{PbL^2}{6} + C_1 L\right] - \frac{P(L-a)^3}{6} = 0$$

or,

$$C_1 = -\frac{Pb}{6L}(L^2 - b^2)\ .$$

Hence,

$$EI\frac{dw}{dx} = \left[\frac{Pbx^2}{2L} - \frac{Pb}{6L}(L^2 - b^2)\right] - \frac{P\langle x-a\rangle^2}{2}$$

$$EIw = \left[\frac{Pbx^3}{6L} - \frac{Pbx}{6L}(L^2 - b^2)\right] - \frac{P\langle x-a\rangle^3}{6}$$

Maximum Deflection:

For w to be maximum, $dw/dx = 0$. Assuming that this happens for $x < a$ we have

$$\frac{Pbx^2}{2L} - \frac{Pb}{6L}(L^2 - b^2) = 0$$

or

$$x = \pm\frac{(L^2 - b^2)^{1/2}}{\sqrt{3}}$$

Clearly $x < 0$ cannot be a solution. Therefore, the maximum deflection is given by

$$EIw_{\max} = \frac{1}{3}\left[\frac{Pb(L^2 - b^2)^{3/2}}{6\sqrt{3}L}\right] - \frac{Pb(L^2 - b^2)^{3/2}}{6\sqrt{3}L}$$

or,

$$w_{\max} = -\frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL}\ .$$

Deflection at load application point

At $x = a$, i.e., at point B, the deflection is

$$EIw_B = \frac{Pba^3}{6L} - \frac{Pba}{6L}(L^2 - b^2) = \frac{Pba}{6L}(a^2 + b^2 - L^2)$$

or

$$w_B = -\frac{Pa^2b^2}{3LEI}$$

Deflection at the mid-point

It is instructive to examine the ratio of $w_{\max}/w(L/2)$. At $x = L/2$

$$EIw(L/2) = \frac{PbL^2}{48} - \frac{Pb}{12}(L^2 - b^2) = -\frac{Pb}{12}\left[\frac{3L^2}{4} - b^2\right]$$

Therefore,

$$\frac{w_{\max}}{w(L/2)} = \frac{4(L^2 - b^2)^{3/2}}{3\sqrt{3}L\left[\frac{3L^2}{4} - b^2\right]} = \frac{4(1 - \frac{b^2}{L^2})^{3/2}}{3\sqrt{3}\left[\frac{3}{4} - \frac{b^2}{L^2}\right]} = \frac{16(1 - k^2)^{3/2}}{3\sqrt{3}\,(3 - 4k^2)}$$

where k = b/L and for a < b we get 0 < k < 0.5. Even when the load is as near as 0.05L from the support, the error in estimating the deflection is only 2.6%. Hence in most of the cases the estimation of maximum deflection may be made fairly accurately with reasonable margin of error by working out deflection at the centre.

Special case of symmetrically applied load

When $a = b = L/2$, for w to be maximum

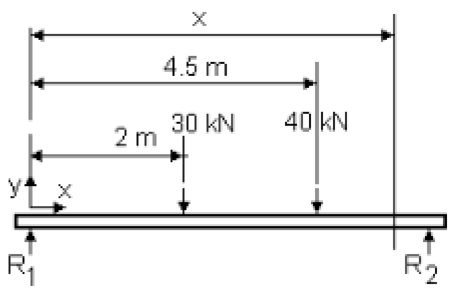
$$x = \frac{[L^2 - (L/2)^2]^{1/2}}{\sqrt{3}} = \frac{L}{2}$$

and the maximum deflection is

$$w_{\max} = -\frac{P(L/2)b[L^2 - (L/2)^2]^{3/2}}{9\sqrt{3}EIL} = -\frac{PL^3}{48EI} = w(L/2)\ .$$

Example 2: Simply Supported Beam with Two Point Loads

The beam shown is 7 m long with an EI value of 200 MN/m^2. Determine the slope and deflection at the middle of the span.



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SOLUTION

First solve the reactions by taking moments about the right end.

$30 \times 5 + 40 \times 2.5 = 7 R_1$ hence $R_1 = 35.71$ kN

$R_2 = 70 - 35.71 = 34.29$ kN

Next write out the bending equation.

$$EI\frac{d^2y}{dx^2} = M = 35710[x] - 30000[x-2] - 40000[x-4.5]$$

Integrate once treating the square bracket as the variable.

$$EI\frac{dy}{dx} = 35710\frac{[x]^2}{2} - 30000\frac{[x-2]^2}{2} - 40000\frac{[x-4.5]^2}{2} + A \ldots..(1)$$

Integrate again

$$EIy = 35710\frac{[x]^3}{6} - 30000\frac{[x-2]^3}{6} - 40000\frac{[x-4.5]^3}{6} + Ax + B \ldots...(2)$$

BOUNDARY CONDITIONS

$x = 0, y = 0$ and $x = 7$ $y = 0$

Using equation 2 and putting $x = 0$ and $y = 0$ we get

$$EI(0) = 35710\frac{[0]^3}{6} - 30000\frac{[0-2]^3}{6} - 40000\frac{[0-4.5]^3}{6} + A(0) + B$$

Ignore any bracket containing a negative value.

$0 = 0 - 0 - 0 + 0 + B$ hence $B = 0$

Using equation 2 again but this time x=7 and y = 0

$$EI(0) = 35710\frac{[7]^3}{6} - 30000\frac{[7-2]^3}{6} - 40000\frac{[7-4.5]^3}{6} + A(7) + 0$$

Evaluate A and $A = -187400$

Moment-Area Method

The moment-area theorem is a method to derive the slope, rotation and deflection of beams and frames. This theorem was developed by Mohr and later stated namely by Charles E. Greene in 1873. This method is advantageous when we solve problems involving beams, especially for those subjected to a series of concentrated loadings or having segments with different moments of inertia. If we draw the moment diagram for the beam and then divided it by the flexural rigidity(EI), the 'M/EI diagram' results by the following:

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx} = \frac{M}{EI} \quad \Rightarrow \theta(x) = \int \frac{M}{EI}dx$$

B. Mohr's Theorems

Theorem 1: The change in slope between any two points on the elastic curve equals the area of the $\frac{M}{EI}$ diagram between these two points.

$$\theta_{AB} = \int_A^B \frac{M}{EI}dx$$

where,

- M = bending moment expression as a function of x
- EI = flexural rigidity
- θ_{AB} = change in slope between points A and B
- A, B = points on the elastic curve

Theorem 2: The vertical deviation of a point A on an elastic curve with respect to the tangent which is extended from another point B equals the moment of the area under the M/EI diagram between those two points (A and B). This moment is computed about point A where the deviation from B to A is to be determined.

$$t_{A/B} = \int_A^B \left(\frac{M}{EI}\right) x\,dx$$

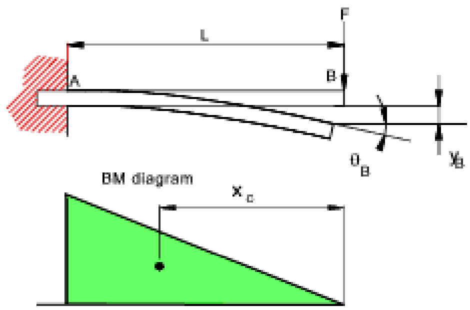
where,

- M = bending moment expression as a function of x
- EI = flexural rigidity
- $t_{A/B}$ = deviation of tangent at point B with respect to the tangent at point A

- A, B = points on the elastic curve

Two simple examples are provide below to illustrate these theorems

Example 1) Determine the deflection and slope of a cantilever as shown..



The bending moment at A = M_A = FL

The area of the bending moment diagram $A_M = F.L^2/2$

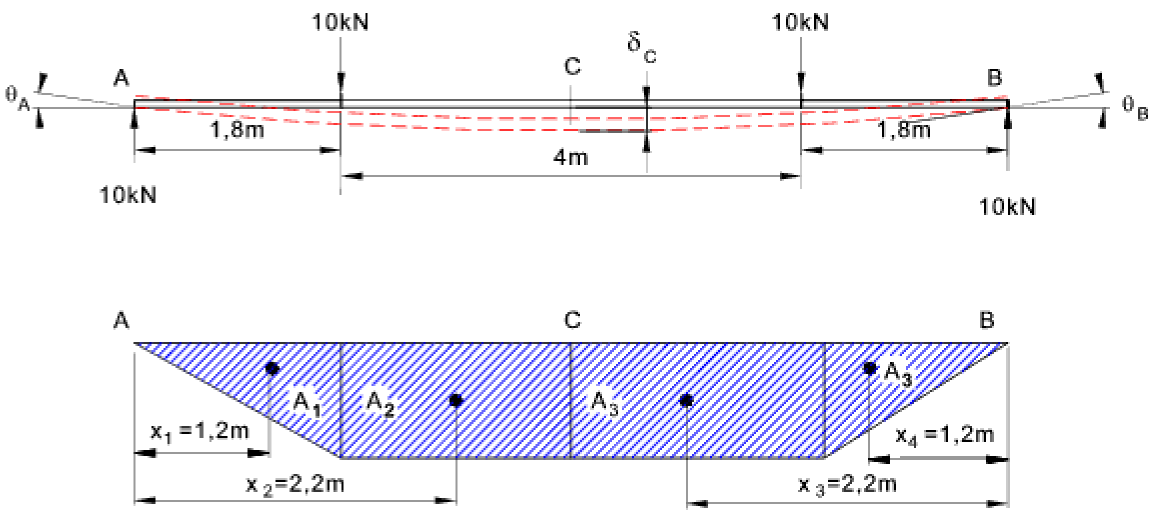
The distance to the centroid of the BM diagram from B = x_c = 2L/3

The deflection of B = $y_b = A_M.x_c/EI = F.L^3/3EI$

The slope at B relative to the tan at A = $\theta_b = A_M/EI = FL^2/2EI$

Example 2) Determine the central deflection and end slopes of the simply supported beam as shown..

E = 210 GPa ......I = 834 cm^4...... EI = 1,7514. $10^6 Nm^2$



Bending Moment Diagram

A_1 = 10.1, 8.1, 8/2 = 16.2kNm

A_2 = 10.1,8.2 = 36kNm

A_2 = 10.1,8.2 = 36kNm

A_1 = 10.1,8.1,8/2 = 16,2kNm

x_1 = Centroid of A_1 = (2/3).1,8 = 1,2
x_2 = Centroid of A_2 = 1,8 + 1 = 2,8
x_3 = Centroid of A_3 = 1,8 + 1 = 2,8
x_4 = Centroid of A_4 = (2/3).1,8 = 1,2

The slope at A is given by the area of the moment diagram between A and C divided by EI.

$\theta_A = (A_1 + A_2) / EI = (16{,}2+36).10^3 / (1{,}7514.\ 10^6)$
$= 0{,}029\text{rads} = 1{,}7$ degrees

The deflection at the centre (C) is equal to the deviation of the point A above a line that is tangent to C.
Moments must therefore be taken about the deviation line at A.

$\delta_C = (A_M.x_M) / EI = (A_1\, x_1 + A_2\, x_2) / EI = 120{,}24.10^3 / (1{,}7514.\ 10^6)$
$= 0.0686\text{m} = 68.6\text{mm}$

Conjugate Beam Method

Conjugate beam is defined as the imaginary beam with the same dimensions (length) as that of the original beam but load at any point on the conjugate beam is equal to the bending moment at that point divided by EI. The conjugate-beam method is a method to derive the slope and displacement of a beam. The conjugate-beam method was developed by H. Müller-Breslau in 1865. Essentially, it requires the same amount of computation as the moment-area theorems to determine a beam's slope or deflection; however, this method relies only on the principles of statics, so its application will be more familiar.

We know the relationship between the load, shear and bending moment in a beam as follows:

(a) The relationship between the load 'w' at a section with the shear force 'V' at that section is

Equation 1: $\frac{dV}{dx} = -w$; and

(b) the relation between the shear force 'V' and the bending moment 'M'at that section is

$\frac{dM}{dx} = V$. Thus, by differentiating this equation we get, Equation 2: $\frac{d^2M}{dx^2} = \frac{dV}{dx} = -w$

The basis for the conjugate-beam method comes from the similarity of the above equations with the slope and deflection equations of the elastic curve.

To show this similarity, these equations are shown below.

Equation 1: $\frac{dV}{dx} = -w$	Equation 2: $\frac{d^2M}{dx^2} = -w$
Equation 3: $\frac{d\theta}{dx} = \frac{M}{EI}$	Equation 4: $\frac{d^2y}{dx^2} = \frac{M}{EI}$

Equation 1 is similar to Equation 3. And Equation 2 is similar to Equation 4. The integral forms of these equations look as follows:

Equation 1: $V = \int -w\,dx$	Equation 2: $M = \int \left(\int -w\,dx \right) dx$
Equation 3: $\theta = \int \left(\dfrac{M}{EI} \right) dx$	Equation 4: $y = \int \left[\int \left(\dfrac{M}{EI} \right) dx \right] dx$

Here the shear V compares with the slope θ, the moment M compares with the displacement y, and the external load w compares with the M/EI diagram.

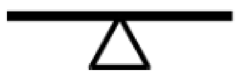
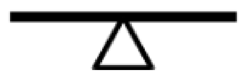
To make use of this comparison we will now consider a beam having the same length as the real beam, but referred here as the "conjugate beam." The conjugate beam is "loaded" with the M/EI diagram derived from the load on the real beam. From the above comparisons, we can state two theorems related to the conjugate beam:

Theorem 1: The slope at a point in the real beam is numerically equal to the shear at the corresponding point in the conjugate beam.

Theorem 2: The displacement of a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam

Supports of the Conjugate Beam:

When drawing the conjugate beam it is important that the shear and moment developed at the supports of the conjugate beam account for the corresponding slope and displacement of the real beam at its supports, a consequence of Theorems 1 and 2.

Real Beam		Conjugate beam	
Fixed Support $y = 0$ $\theta = 0$		**Free End** $\overline{M} = 0$ $\overline{V} = 0$	
Free End $y \neq 0$ $\theta \neq 0$		**Fixed Support** $\overline{M} \neq 0$ $\overline{V} \neq 0$	
Hinged support $y = 0$ $\theta \neq 0$		**Hinged support** $\overline{M} = 0$ $\overline{V} \neq 0$	
Middle support $y = 0$ θ continuous		**Middle hinge** $\overline{M} = 0$ $\overline{V}$ continuous	
Middle hinge $y =$ continuous θ discontinuous		**Middle support** $\overline{M}$ continous $\overline{V}$ discontinuous	

For example, as shown above, a pin or roller support at the end of the real beam provides zero displacement, but a non zero slope. Consequently, from Theorems 1 and 2, the conjugate beam must be supported by a pin or a roller, since this support has zero moment but has a shear or end reaction. When the real beam is fixed supported, both the slope and displacement are zero. Here the conjugate beam has a free end, since at this end there is zero shear and zero moment. Corresponding real and conjugate supports are shown below. Note that, as a rule, neglecting axial forces, statically determinate real beams have statically determinate conjugate beams; and statically indeterminate real beams have unstable conjugate beams. Although this occurs, the M/EI loading will provide the necessary "equilibrium" to hold the conjugate beam stable.

Some Examples of Conjugate Beams:

	Real beam	Conjugate beam
Simple beam		
Cantilever beam		
Left-end Overhanging beam		
Both-end overhanging beam		
Gerber's beam (2 span)		
Gerber's beam (3 span)		

Analysis Proceedure:

The following procedure provides a method that may be used to determine the displacement and slope at a point on the elastic curve of a beam using the conjugate-beam method.

Conjugate beam

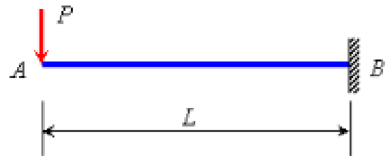
- This beam has the same length as the real beam and has corresponding supports as listed above.
- In general, if the real support allows a slope, the conjugate support must develop shear; and if the real support allows a displacement, the conjugate support must develop a moment.
- The conjugate beam is loaded with the real beam's M/EI diagram. This loading is assumed to be distributed over the conjugate beam and is directed upward when M/EI is positive and downward when M/EI is negative. In other words, the loading always acts away from the beam.

Equilibrium

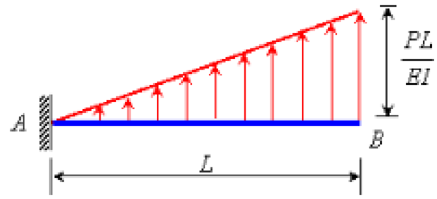
- Using the equations of statics, determine the reactions at the conjugate beams supports.
- Section the conjugate beam at the point where the slope θ and displacement Δ of the real beam are to be determined. At the section show the unknown shear V' and M' equal to θ and Δ, respectively, for the real beam. In particular, if these values are positive, and slope is counterclockwise and the displacement is upward.

Example 1:

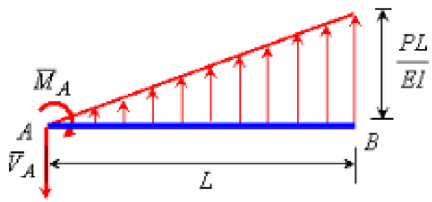
Determine the slope and deflection of point *A* of the of a cantilever beam *AB* of length *L* and uniform flexural rigidity *EI.* A concentrated force *P* is applied at the free end *of* beam.



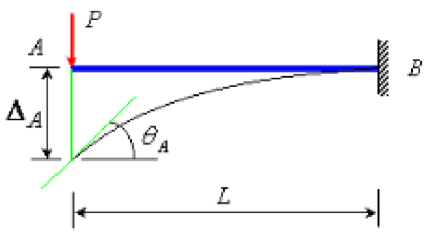
(a) A cantilever beam (actual beam)



(b) Conjugate beam (additional beam) corresponding to the actual beam



(c) Free-body diagram for the conjugate beam



(d) Deflections of the cantilever beam (actual beam)

Solution: The conjugate beam of the actual beam is shown in Figure (b). A linearly varying distributed upward elastic load with intensity equal to zero at A and equal to PL/EI at B. The free-body diagram for the conjugate beam is shown in Figure 8(c). The reactions at A of the conjugate beam are given by

$$\overline{V}_A = \frac{1}{2} \times L \times \frac{PL}{EI} = \frac{PL^2}{2EI} \left(\downarrow \right)$$
$$\overline{M}_A = \left(\frac{1}{2} \times L \times \frac{PL}{EI} \right) \times \frac{2L}{3} = \frac{PL^3}{3EI} \left(\curvearrowright \right)$$

The slope at A, and the deflection at the free end A of the actual beam in Figure (d) are respectively,

equal to the “shearing force” and the “bending moment” at the fixed end A of the conjugate beam in

Figure (c).

$$\theta_A = \frac{PL^2}{2EI}\left(\circlearrowright\right)$$

$$\Delta_A = \frac{PL^3}{3EI}\left(\downarrow\right)$$

UNIT – IV

DEFLECTION OF BEAMS

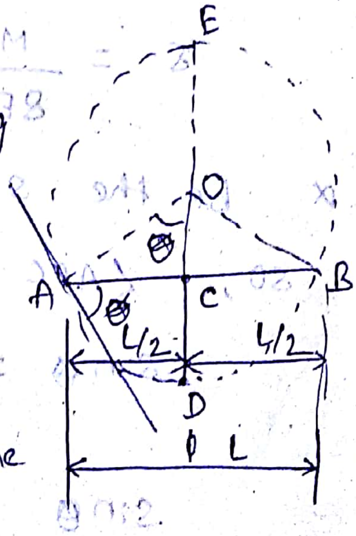
It is observed that when a beam (on a cantilever is subjected to some type of loading it deflects from its initial /original position. The amount of deflection depends upon its cross-section and bending moment. It is necessary to determine deflections to verify whether they are within tolerable limits. If the deflections exceeding the tolerable limits, result will be failed.

If the member is subjected to a uniform bending moment M, the radius of curvature of the deflected form of the member is given by

$$\frac{M}{I} = \frac{E}{R}.$$

Member bending into circular arc :

A member AB of span L subjected to a uniform bending moment M, so that the member is bent into a circular shape.



Let R be the radius of the bent form of the member.

Let the deflection at the centre of the span be CD $= \delta$.

But $DC \cdot CE = AC \cdot CB$

$$\delta \times (2R - \delta) = \frac{l}{2} \times \frac{l}{2}$$

$$= \frac{l^2}{4}$$

$$2R\delta - \delta^2 = \frac{l^2}{4}$$

For a particular beam, the deflection δ being a small quantity, hence δ^2 can be neglected.

$$2R\delta = \frac{l^2}{4}$$

$$\therefore \delta = \frac{l^2}{8R} \Rightarrow \frac{1}{R} = \frac{8\delta}{l^2}$$

But $\frac{M}{I} = \frac{E}{R}$

$$\frac{1}{R} = \frac{M}{EI}$$

$$\frac{8\delta}{l^2} = \frac{M}{EI}$$

$$\delta = \frac{Ml^2}{8EI}$$

let α be the slope at the end.

so, $\angle AOC = \theta$

$$\sin\theta = \frac{AC}{OA} = \frac{l/2}{R} = \frac{l}{2R}$$

$$\sin\theta = \frac{Ml}{2EI} \quad \left[\because \frac{1}{R} = \frac{M}{EI}\right]$$

$\sin\theta = \theta$ radians for small values of θ.

$$\boxed{\therefore \theta = \frac{Ml}{2EI}} \text{ radians}$$

Relation between slope, deflection and radius of curvature:

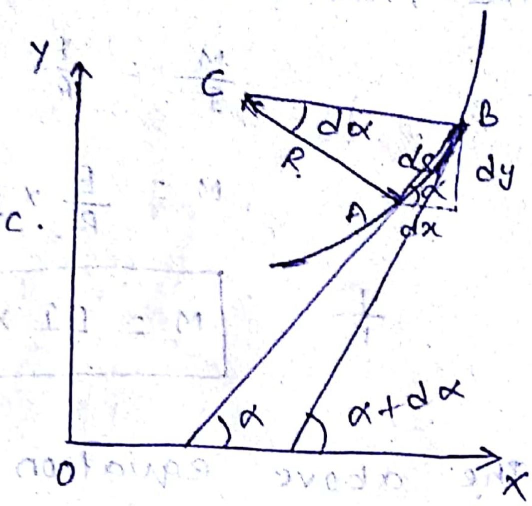


Fig. shows a small portion AB of a beam bent into an arc.

Let, ds - length of beam AB,

C - centre of arc (into which the beam has been bent).

α - Angle which the tangent at A makes with x-x axis.

$\alpha + d\alpha$ - Angle which the tangent at B makes with x-x axis.

$\angle ACB = d\alpha$, $ds = R\,d\alpha$ $\left[\because \sin d\alpha = \frac{ds}{R}\quad d\alpha = \frac{ds}{R}\quad ds = R\,d\alpha\right]$.

$$R = \frac{ds}{d\alpha}$$

$$= \frac{dx}{d\alpha} \quad [\text{Assuming } ds = dx]$$

$$\therefore \frac{1}{R} = \frac{d\alpha}{dx}.$$

If the coordinates of point A are x & y, then

$$\tan\alpha = \frac{dy}{dx} \quad (\text{or}) \quad \alpha = \frac{dy}{dx} \quad \left[\because \alpha \text{ is small } \tan\alpha = \alpha\right]$$

$$\alpha = \frac{dy}{dx}$$

differentiating the above equation w.r.to x, we get

$$\frac{d\alpha}{dx} = \frac{d^2y}{dx^2}. \qquad \because \frac{1}{R} = \frac{d\alpha}{dx}$$

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

Also, we know that

$$\frac{M}{I} = \frac{E}{R} \quad \text{con}$$

$$M = \frac{E}{R} \times I \qquad \left[\because \frac{1}{R} = \frac{d^2y}{dx^2}\right]$$

$$\boxed{M = EI \times \frac{d^2y}{dx^2}}$$

The above equation is called the differential equation of flexure of a beam. It must be valid only when the slopes are small.

differentiating the above equ. w.r.t x, we get

$$\frac{dM}{dx} = EI \times \frac{d^3y}{dx^3}$$

$$F = EI \times \frac{d^3y}{dx^3}$$ F - shearing force

Again differentiating w.r.t x

$$\frac{dF}{dx} = EI \times \frac{d^4y}{dx^4}$$

$$w = EI \times \frac{d^4y}{dx^4}$$

Where,

y - deflection

$\frac{dy}{dx}$ - slope.

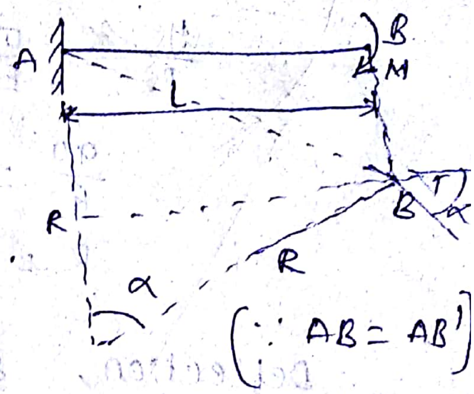
Bending moment $(M) = EI \frac{d^2y}{dx^2}$

Shearing force $(F) = EI \frac{d^3y}{dx^3}$

The rate of loading $(w) = EI \frac{d^4y}{dx^4}$

③

Pblm A cantilever of length L is subjected to a couple M at its free end. Find the slope and deflection of the end.



Sol

$$\frac{1}{R} = \frac{M}{EI}$$

Let α - slope at the free end B.

$$\alpha = \sin\alpha = \frac{AB'}{R} = \frac{L}{R}$$

$(\because AB = AB')$

$$\alpha = \frac{ML}{EI}$$

By the property of circles

$$\delta \times (2R - \delta) = L^2$$

$$2R\delta - \delta^2 = L^2$$

$$2R\delta = L^2 \qquad \delta^2 - \text{negligible}$$

$$\delta = \frac{L^2}{2R}$$

$$\delta = \frac{ML^2}{2EI}$$

Pblm A cantilever of length 1.25 m is subjected to a couple M_0 at the free end. The longitudinal strain at the top surface is 0.0015 and the distance of the top surface of the cantilever from the neutral layer is 90mm. Find the radius of the neutral layer and the vertical deflection at the end of the cantilever.

Sol $L = 1.25\,m = 1250\,mm$

$BM = M_0$

$$\frac{\sigma}{y} = \frac{E}{R}$$

$\because \varepsilon = 0.0015$;

$\frac{\sigma}{E} = \varepsilon$

$$\frac{y}{R} = \frac{\sigma}{E} = 0.0015$$

$$\frac{90}{R} = 0.0015$$

$$R = 60000 \text{ mm} = 60 \text{ m}$$

Deflection, $\delta = \frac{L^2}{2R}$

$$\delta = \frac{(1250)^2}{2 \times 60000} = 13.02 \text{ mm}.$$

Slope and Deflection at a section:

The important methods used for finding out the slope and deflection at a section in a loaded beam are discussed as follows:

1. Double integration method
2. Moment area method
3. Macaulay's method.

The first two methods are suitable for a single load, whereas the last one is suitable for several loads.

Double integration method

Cantilevers:

Case-1: Cantilever beam - concentrated load at free end

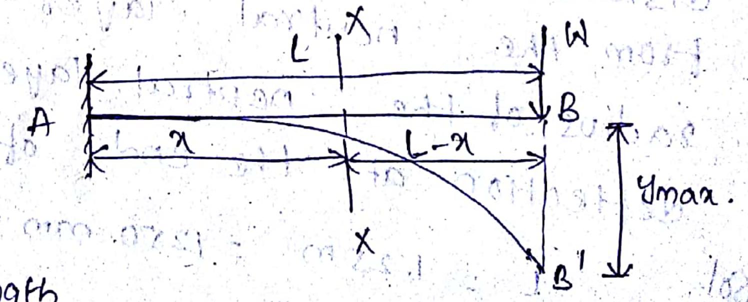


Fig. shows a cantilever beam AB of uniform ~~strengt~~ section and of length L fixed at the end A and free at B

Let a concentrated load W be applied at the free end B.

let I - MOI of the section of the cantilever about the N.A.

consider a section XX at a distance x from the fixed end A.

BM at the section is

$$EI \therefore M_x = -W(L-x)$$

$$EI \frac{d^2y}{dx^2} = -W(L-x)$$

Integrating, we get

$$EI \frac{dy}{dx} = -W\left(Lx - \frac{x^2}{2}\right) + C_1$$

where C_1 - integration constant.

At A

$x = 0$, $\frac{dy}{dx} = 0$.

$$EI \times 0 = -W(0) + C_1$$

$$C_1 = 0$$

$$\therefore EI \frac{dy}{dx} = -W\left(Lx - \frac{x^2}{2}\right) \longrightarrow (1) \text{ slope equation.}$$

At B $x = L$, slope at B $= \theta_B$

$$\theta_B = \frac{dy}{dx} = -\frac{1}{EI} W \times \left(L \times L - \frac{L^2}{2}\right)$$

$$\boxed{\theta_B = -\frac{wL^2}{2EI}} \longrightarrow (a)$$

To get deflection, integrating equ (1), we get,

$$EI \int \frac{dy}{dx} = -w \int \left(Lx - \frac{x^2}{2}\right) dx$$

$$EI\, y = -w \times \left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + C_2$$

At A

$x = 0$, $y = 0$

$$EI \times 0 = -w \times 0 + C_2$$

$$C_2 = 0$$

$$\therefore EI\, y = -w\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) \longrightarrow (2) \text{ Deflection equation}$$

Deflection at B,

$x = L$, we get

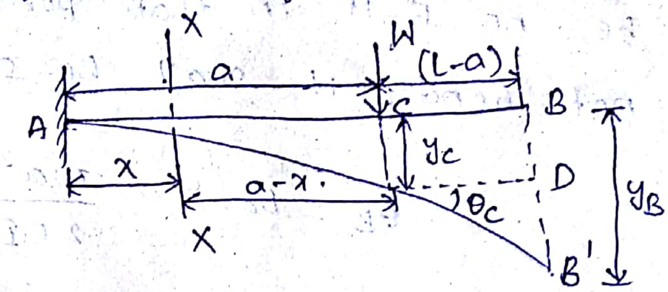
$$y_B = -\frac{1}{EI} \times w \times \left(L \times \frac{L^2}{2} - \frac{L^3}{6}\right)$$

$$= -\frac{w}{EI} \times \frac{L^3}{3}$$

$$\boxed{y_B = \frac{wL^3}{3EI}} \text{ (downward deflection)} \longrightarrow (b)$$

The equ's (a) & (b) give maximum values of slope and deflection at the free end.

Case-2: cantilever of length L carrying a concentrated load W at distance a from the fixed end:



consider a section XX at a distance x from the fixed end A.

$$M_x = -w(a-x)$$

$$EI\frac{d^2y}{dx^2} = -w(a-x)$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -w\left(ax - \frac{x^2}{2}\right) + C_1$$

At A

$x = 0$, $\frac{dy}{dx} = 0$

$$EI \times 0 = -w \times 0 + C_1$$

$$C_1 = 0$$

$$\therefore EI\frac{dy}{dx} = -w\left(ax - \frac{x^2}{2}\right) \longrightarrow (1) \text{ slope equation}$$

slope at C,

$x = a$, we get

$$\theta_c = \frac{dy}{dx} = \frac{-w}{EI}\left(a \times a - \frac{a^2}{2}\right)$$

$$\boxed{\theta_c = -\frac{wa^2}{2EI}}$$

As there is no load on the portion BC; there will be no B.M in that portion and the portion will not bend, it shall be straight.

$$\theta_B = \theta_C = \frac{-wa^2}{2EI}$$

To get deflection, integrating equ (1), we get

$$EIy = -w\left(a \times \frac{x^2}{2} - \frac{x^3}{6}\right) + C_2$$

<u>At A</u>

$x = 0,\ y = 0$

$EI \times 0 = -w \times 0 + C_2$

$C_2 = 0$

$$\therefore\ EIy = -w\left(a \cdot \frac{x^2}{2} - \frac{x^3}{6}\right) \longrightarrow (2) \text{ Deflection equation}$$

<u>Deflection at C,</u>

$x = a$, we get

$$y_c = \frac{-w}{EI}\left(a \times \frac{a^2}{2} - \frac{a^3}{6}\right)$$

$$= \frac{-wa^3}{3EI}$$

$$y_c = \frac{wa^3}{3EI} \quad \text{(Downward deflection)}$$

But $y_c = BD$, $B'D = DC' \tan\theta_C$

$= BC\,\theta_C$ $\quad [\because DC' = BC]$

$$B'D = (l - a) \times \left(\frac{-wa^2}{2EI}\right)$$

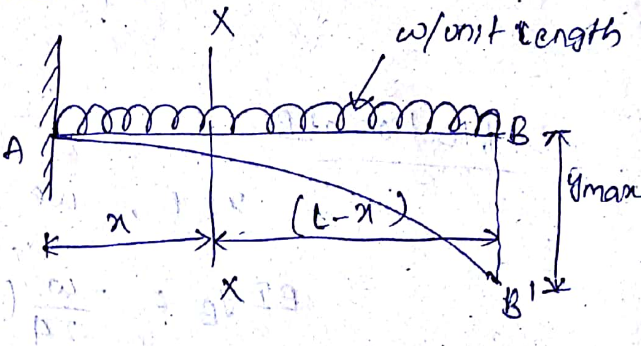
But, $y_B = BB' = BD + B'D$

$$= \frac{-wa^3}{3EI} - \frac{wa^2}{2EI}(l-a)$$

$$y_B = \frac{wa^3}{3EI} + \frac{wa^2}{2EI}(l-a) \quad \text{(Downward deflection)}$$

<u>case-3</u>: cantilever of length L carrying UDL w per unit run over whole length.

consider a section xx at a distance x from fixed end A



$$M_x = \frac{-w(l-x)^2}{2}$$

$$EI\frac{d^2y}{dx^2} = \frac{-w}{2}(L-x)^2$$

Integrating, we get

$$EI\frac{dy}{dx} = \frac{w}{6}(L-x)^3 + C_1$$

<u>At A</u> $x = 0, \quad \frac{dy}{dx} = 0$

$$EI \times 0 = \frac{w}{6} \times L^3 + C_1$$

$$C_1 = -\frac{wL^3}{6}$$

$$EI\frac{dy}{dx} = \frac{w}{6} \times (L-x)^3 - \frac{wL^3}{6} \longrightarrow (1) \quad \text{slope equation}$$

<u>slope at B</u>

$x = L$, we have

$$EI\theta_B = \frac{dy}{dx} = \frac{w}{6} \times (L-L)^3 - \frac{wL^3}{6} = \frac{-wL^3}{6}$$

$$\boxed{\theta_B = -\frac{wL^3}{6EI} = \frac{-WL^2}{6EI}} \quad (W = w \times L)$$

To get deflection, integrating equ. ①, we get

$$EIy = \frac{-w}{24}(L-x)^4 - \frac{wL^3}{6}x + C_2$$

At A

$x = 0, \quad y = 0.$

$$EI \times 0 = \frac{-wL^4}{24} + C_2$$

$$C_2 = \frac{wL^4}{24}$$

$$EIy = \frac{-w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \longrightarrow ②$$

Deflection equ.

Deflection at B

$x = L$, we get

$$EIy_B = \frac{-w}{24}(L-L)^4 - \frac{wL^3}{6} \times L + \frac{wL^4}{24}$$

$$= -\frac{wL^4}{6} + \frac{wL^4}{24} = \frac{-wL^4}{8}.$$

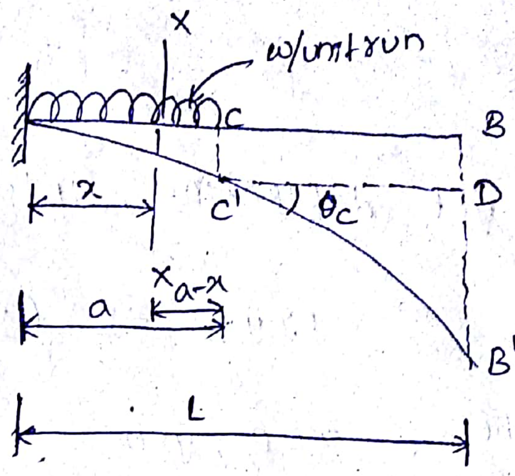
$$\therefore y_B = \frac{-wL^4}{8EI}$$

$[W = w \times L]$

$$y_B = \frac{wL^4}{8EI} = \frac{WL^3}{8EI} \quad \text{(Downward deflection)}$$

The equ. θ_B & y_B give slope & deflection at B.
which are maximum.

Case-4 : Cantilever of length 'L' carrying UDL of w per unit run for a distance a from the fixed end.



Consider a section XX at a distance x from fixed end A.

$$M_x = -\frac{w(a-x)^2}{2}$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = \frac{-w(a-x)^2}{2}$$

Integrating on both sides

$$EI\frac{dy}{dx} = +\frac{w}{2}\left[\frac{(a-x)^3}{3}\right] + C_1$$

When At A

$$x = 0, \quad \frac{dy}{dx} = 0$$

$$EI \times 0 = \frac{w}{2} \times \frac{(a-0)^3}{3} + C_1$$

$$-\frac{wa^3}{6} = C_1$$

Hence, $$EI.\frac{dy}{dx} = \frac{w}{2} \times \frac{(a-x)^3}{3} - \frac{wa^3}{6} \longrightarrow (1)$$

slope equation

Slope at C

$x = a$, we get

$$\theta_C = \frac{dy}{dx} = \frac{-wa^3}{6EI}$$

Since portion BC is not loaded, it does not bend and remains straight,

$$\therefore \theta_B = \theta_C = \frac{-wa^3}{6EI}$$

$$= \frac{-Wa^2}{6EI} \qquad [\because W = wa]$$

To get deflection, integrating equ ①, we get

$$EIy = -\frac{w}{2}\frac{(a-x)^4}{12} - \frac{wa^3}{6}x + C_2$$

when, $x=0$, $y=0$

$$EI \times 0 = -\frac{w}{2} \times \frac{(a-0)^4}{12} - \frac{wa^3}{6} \times 0 + C_2$$

$$C_2 = \frac{wa^4}{24}$$

$$EIy = -\frac{w}{2}\frac{(a-x)^4}{12} - \frac{wa^3}{6}x + \frac{wa^4}{24} \longrightarrow ②$$

Deflection equation.

<u>Deflection at c,</u>

$x = a$, we get

$$EIy_c = -\frac{wa^4}{6} + \frac{wa^4}{24}$$

$$= \frac{-wa^4}{8}$$

$$y_c = \frac{-wa^4}{8EI} = -\frac{Wa^3}{8EI} \qquad (\because W = w.a)$$

$$cc' = BD = \frac{-wa^4}{8EI}.$$

But, $B'D = c'D \tan\theta_c = BC \times \theta_c$

$$= (L-a) \times \left(-\frac{wa^3}{6EI}\right)$$

$$y_B = BD + B'D$$

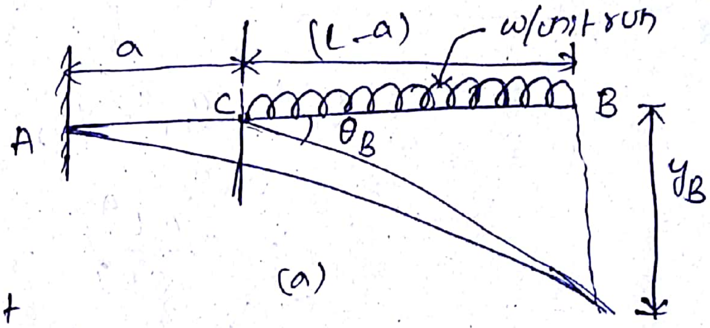
$$= \frac{-wa^4}{8EI} + (l-a) \times \left(-\frac{wa^3}{6EI}\right)$$

$$= -\left[\frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l-a)\right]$$

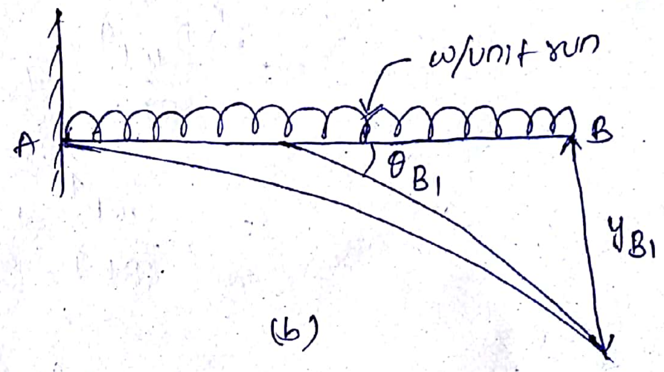
$$y_B = \frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l-a) \quad \text{(Downward deflection)}$$

$$= \frac{Wa^3}{8EI} + \frac{Wa^2}{6EI}(l-a)$$

<u>case-5</u>: cantilever of length l carrying a UDL of w per unit run on a part of span from the free end.



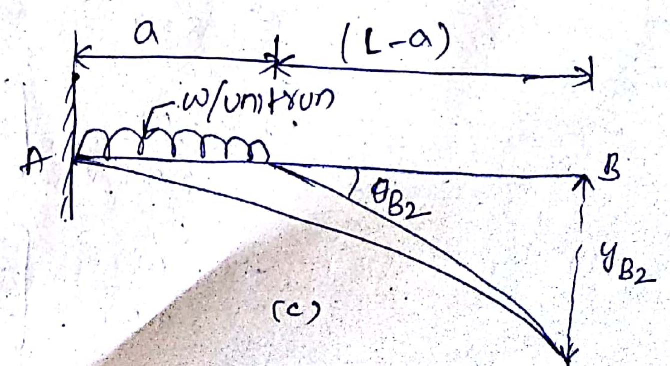
It is obvious from fig. (a, b, c) that to get result in case (a) take the differences of result in case (b) and case (c), thus:



$$\theta_B = \theta_{B_1} - \theta_{B_2}$$

$$y_B = y_{B_1} - y_{B_2}$$

But, from the previous articles, we have



$$\theta_{B_1} = \frac{wl^3}{6EI}, \quad y_{B_1} = \frac{-wl^4}{8EI}$$

$$\theta_{B_2} = \frac{wa^3}{6EI}, \quad \cancel{y_{B_2}} = -$$

E & I - constant.

$$y_{B_2} = -\left[\frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l-a)\right]$$

<u>slope at B</u>

$$\theta_B = \theta_{B_1} - \theta_{B_2}$$

$$= \frac{-wl^3}{6EI} - \left(-\frac{wa^3}{6EI}\right)$$

$$= \frac{-wl^3}{6EI} + \frac{wa^3}{6EI} = \frac{-w}{6EI}(l^3 - a^3)$$

$$\theta_B = \frac{-w}{6EI}(l^3 - a^3)$$

<u>Deflection at B</u>

$$y_B = y_{B_1} - y_{B_2}$$

$$= \frac{-wl^4}{8EI} - \left[-\left\{\frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l-a)\right\}\right]$$

$$= \frac{-wl^4}{8EI} + \frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l-a)$$

$$= \frac{-wl^4}{8EI} + \frac{wa^4}{8EI} + \frac{wa^3 l}{6EI} - \frac{wa^4}{6EI}$$

$$= \frac{-w}{8EI}(l^4 - a^4) - \frac{wa^3}{6EI}(l-a)$$

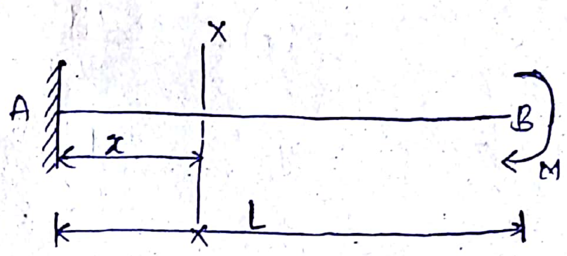
$$= \frac{-w}{24EI}\left[3(l^4 - a^4) - 4a^3(l-a)\right]$$

$$= \frac{-w}{24EI}\left[3l^4 - 3a^4 - 4a^3 l + 4a^4\right]$$

$$= \frac{-w}{24EI}\left(3L^4 - 4La^3 + a^4\right)$$

$$y_B = \frac{w}{24EI}\left(3L^4 - 4La^3 + a^4\right)$$ Downward deflection

Case-6: cantilever of length L with a moment applied at the free end:



$$M_x = -M,$$

$$EI\frac{d^2y}{dx^2} = -M$$

Integrating on both sides, we have

$$EI\frac{dy}{dx} = -Mx + C_1$$

At A

$$x = 0, \quad \frac{dy}{dx} = 0$$

$$EI \times 0 = -M \times 0 + C_1$$

$$C_1 = 0$$

Hence, $EI\frac{dy}{dx} = -Mx \longrightarrow (1)$ slope equation.

Slope at B

$$x = L$$

$$\theta_B = \frac{dy}{dx} = \frac{-ML}{EI}$$

$$EIy = -M \cdot \frac{x^2}{2} + C_2$$

At A

$$x = 0, \quad y = 0$$

$$C_2 = 0$$

Hence, $EIy = -M \cdot \frac{x^2}{2} \longrightarrow (2)$ Deflection equ.

Deflection at B

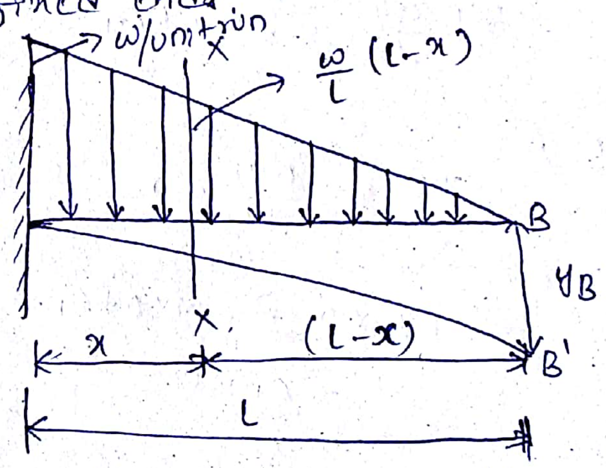
$$x = L$$

$$EIy_B = -\frac{ML^2}{2}$$

$$y_B = -\frac{ML^2}{2EI}$$

$$y_B = \frac{ML^2}{2EI} \quad \text{(Downward deflection)}$$

Case-7: cantilever of length L carrying a UVL when intensity varies uniformly from zero at the free end to w per unit run at the fixed ends.



consider a section xx at a distance x from the fixed end.

Intensity of loading at the section xx $= \frac{w}{L}(L-x)$ per unit run

BM at the section xx

$$M_x = -\frac{1}{2}(L-x) \times \frac{w}{L}(L-x) \times \left(\frac{L-x}{3}\right)$$

$$= -\frac{w(L-x)^3}{6L}$$

$$EI\frac{d^2y}{dx^2} = -\frac{w(L-x)^3}{6L}$$

Integrating on both sides, we get

$$EI\frac{dy}{dx} = \frac{\omega(L-x)^4}{24L} + C_1$$

At A

$$x = 0, \quad \frac{dy}{dx} = 0.$$

$$C_1 = -\frac{\omega L^3}{24}$$

$$EI\frac{dy}{dx} = \frac{\omega(L-x)^4}{24L} - \frac{\omega L^3}{24} \longrightarrow (1) \quad \text{slope equation}$$

Slope at B,

$$x = L$$

$$\theta_B = \frac{dy}{dx} = \frac{-\omega L^3}{24EI}$$

For y   Integrating again, we get

$$EIy = \frac{-\omega(L-x)^5}{120L}$$

$$= -\frac{\omega L^3}{24}x + C_2$$

At A

$$x = 0, \quad y = 0$$

$$C_2 = \frac{\omega L^4}{120}$$

$$EIy = -\frac{\omega(L-x)^5}{120L} - \frac{\omega L^3}{24}x + \frac{\omega L^4}{120}$$

Deflection at B

$$x = L$$

$$EIy_B = \frac{-\omega L^4}{24} + \frac{\omega L^4}{120} = -\frac{\omega L^4}{30}$$

$$y_B = -\frac{\omega L^4}{30EI}$$

$$y_B = \frac{\omega L^4}{30EI} \quad \text{Downward deflection.}$$

Simply supported beams

Case-1 : Simply supported beam of span L carrying a point load at mid span.

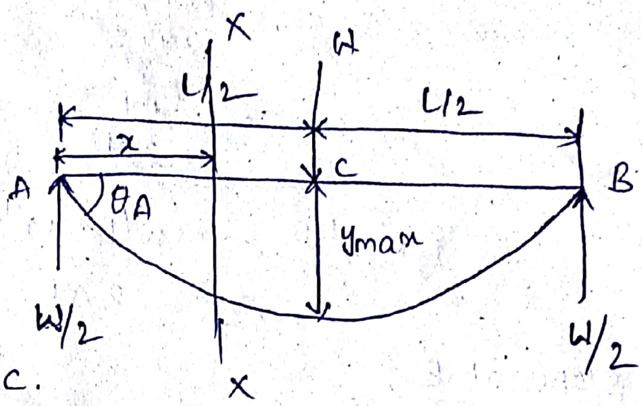


Fig. shows a simply supported beam AB of span L carrying a point load W at the mid span C.

Since the load is symmetrically applied the maximum deflection (y_{max}) will occur at mid span.

Each vertical reaction equals $\frac{W}{2}$.

Consider the left half AC of the span. The B.M at any section XX in AC distant x from A is given by

$$EI\frac{d^2y}{dx^2} = \frac{W}{2}x$$

Integrating, we get

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} + C_1$$

When, $x = \frac{L}{2}$, $\frac{dy}{dx} = 0$.

$$0 = \frac{W}{4}(L/2)^2 + C_1$$

$$C_1 = \frac{WL^2}{16}$$

Hence, $EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL^2}{16}$ ⟶ ① slope equation

Slope at A

$$x = 0$$

$$\theta_A = \frac{dy}{dx} = -\frac{wL^2}{16EI}$$

$$\theta_A = -\frac{wL^2}{16EI}$$

Integrating the slope equation, we get

$$EIy = \frac{wx^3}{12} - \frac{wL^2}{16}x + C_2$$

$$x = 0,\ y = 0$$

$$\therefore\ C_2 = 0$$

$$EIy = \frac{wx^3}{12} - \frac{wL^2}{16}x \longrightarrow (2) \text{ Deflection equ.}$$

Deflection at C :

$$x = L/2$$

$$EIy_C = \frac{W \times (L/2)^3}{12} - \frac{wL^2}{16}(L/2)$$

$$= \frac{wL^3}{96} - \frac{wL^3}{32} = \frac{-wL^3}{48}$$

$$y_C = \frac{-wL^3}{48EI}$$

$$y_C = \frac{WL^3}{48EI} \text{ (downward deflection)}$$